

Hydrodynamic Solver with “Realistic” Equation of State and the Radiation Pressure Contribution

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Summary

- To construct a **conservative** numerical scheme for solving the high-temperature plasma motion, one needs to solve the numerical fluxes across the control volume faces.
- To construct the energy and momentum fluxes, the pressure should be solved. For a given substance, the pressure can be expressed in terms of the conserved mass density and the internal energy density (the latter being the difference between the conserved total energy density and the kinetic energy density). To relate the pressure to the mass density and the internal energy density, one needs to apply the Equation Of State (EOS)
- An approximation of an ideal gas with a constant polytropic index does not provide a good description of the thermodynamic parameters of real high-Z plasmas (such as a Xenon plasma of a the temperature of ~100 eV). What is even more significant, the numerical schemes and algorithms developed under a (highly restrictive) assumption of a constant polytropic index cannot be extended for the case when the EOS: (1) is provided by more complicated analytical or semi-analytical physical model; (2) is tabulated; or (3) is supplied by some black-box-like module in the framework code, with a priori unknown properties (except for the assumed general thermodynamic consistency).
- As an example of the “realistic” EOS we use a numerical model of an ionization equilibrium in an ideal plasma. The ionization equilibrium (the ion populations) is solved using a statistical sum method, or, equivalently, by solving a coupled system of the Saha equilibrium equations.
- To construct a stable and accurate second-order Godunov numerical scheme we apply the limited reconstruction procedure in conjunction with the exact Riemann solver for a polytropic gas with $\gamma=5/3$. We apply a known ‘relaxation’ procedure to benefit from both the simplicity of the constant-gamma exact Riemann solver and the accuracy of the realistic equation of state. This approach is based on the assumption that the exact internal energy density can be split for: (1) the contribution from the translational motion, $P(\gamma-1)$, with P being the exact pressure found from the EOS; and (2) the contribution from the ionization energy density, which is assumed to be decoupled from the translational temperature within the time step.
- In order to numerically solve the coupled system of the radiation hydrodynamics, in case the radiation transport and the hydrodynamic advection are handled by different models and codes, a matter of a crucial choice is if the (evolving) radiation pressure is included into the hydrodynamic model or as the momentum-energy sources for the plasma state vector predicted by the radiation transport model. We advocate the first of the two alternatives.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad \text{Hydro equations}$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + P \mathbf{I}) = 0,$$

$$\frac{\partial (\rho \frac{u^2}{2} + E)}{\partial t} + \nabla \cdot (\mathbf{u} (\rho \frac{u^2}{2} + E + P)) = 0,$$

$$P = P_{EOS}(E, \rho)$$

Relaxation Scheme. 1. Assume the energy equation to be a sum of two conservation laws:

$$\frac{\partial (E^*)}{\partial t} + \nabla \cdot (\mathbf{u} E^*) = 0,$$

$$\frac{\partial (\rho \frac{u^2}{2} + \frac{P}{\gamma-1})}{\partial t} + \nabla \cdot (\mathbf{u} (\rho \frac{u^2}{2} + \frac{P}{\gamma-1} + P)) = 0,$$

Relaxation Scheme. 2. At the beginning of the time step define the ionization energy density as the difference between the internal energy density and the contribution from the translational energy:

$$E^* = E - \frac{P}{\gamma-1},$$

Relaxation Scheme. 3. Advance a numerical solution through the time step. At the end of time step recover the internal energy as the total of the advected ionization energy and the evolved contribution from the translational energy:

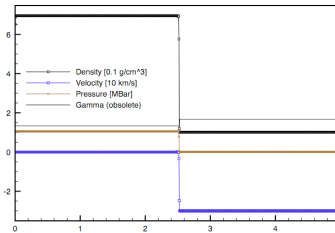
$$E = E^* + \frac{P}{\gamma-1}.$$

Relaxation Scheme. 4. Recover the updated pressure from the updated internal energy density and updated mass density using the EOS:

$$P = P_{EOS}(E, \rho)$$

Test result for the relaxation scheme

- The test result for the piston shock wave in a high-pressure xenon. The piston speed is 30 km/s
- No spurious oscillations behind the shock wave front (there are some in the density distribution unless the relaxation scheme is used)
- The compression ratio in the shock wave (>7) is much greater than the maximum compression ratio (4) in the ideal gas of a constant polytropic index of 5/3



How can we split by physical effects the system of the equations describing the radiation transport in the grey diffusion approximation?

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + (P + P_{RAD}) \mathbf{I}) = 0,$$

$$\frac{\partial (\rho \frac{u^2}{2} + E + E_{RAD})}{\partial t} + \nabla \cdot (\mathbf{u} (\rho \frac{u^2}{2} + E + E_{RAD} + P + P_{RAD})) = 0,$$

$$\frac{\partial E_{RAD}}{\partial t} + \nabla \cdot (\mathbf{u} E_{RAD}) + P_{RAD} \nabla \cdot \mathbf{u} = \nabla \cdot (D \nabla E_{RAD}) + \alpha (a T^4 - E_{RAD}),$$

$$P = P_{EOS}(E, \rho), \quad P_{RAD} = \frac{E_{RAD}}{3}$$

The suggestion is to keep the evolving radiation pressure and energy in the hydrodynamic evolutionary operator:

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + (P + P_{RAD}) \mathbf{I}) = 0,$$

$$\frac{\partial (\rho \frac{u^2}{2} + E + E_{RAD})}{\partial t} + \nabla \cdot (\mathbf{u} (\rho \frac{u^2}{2} + E + E_{RAD} + P + P_{RAD})) = 0,$$

$$\frac{\partial E_{RAD}}{\partial t} + \nabla \cdot (\mathbf{u} E_{RAD}) + P_{RAD} \nabla \cdot \mathbf{u} = 0$$

For the latter system the maximum propagation speed of small perturbations seems to be the most reliable:

$$C_{\max} \approx |\mathbf{u}| + \sqrt{\left(\frac{5}{3}P + \frac{4}{3}P_{RAD}\right) / \rho} \geq |\mathbf{u}| + \sqrt{\frac{5}{3}P / \rho}$$

Otherwise, the numerical diffusion in the hydrodynamic solver may be insufficient and/or the Courant stability criterion for the explicit numerical scheme may be wrong, as long as C_{\max} does not account for the contribution from the radiation pressure.