Sweepless Time-Dependent Transport Calculations using the Staggered Block Jacobi Method

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Abstract

The Staggered Block Jacobi method for time-dependent transport problems is a sweepless and inherently parallel transport method. It is highly accurate in thick-diffusive problems and unconditionally stable when combined with the lumped linear discontinuous finite element spatial discretization.

Introduction

The linear Boltzmann transport equation describes a rarefied field of neutral particles streaming through and interacting with a background material.

The transport equation:

Time derivative of flux — Collision Term — Source

$$\frac{1}{v}\frac{\partial}{\partial t}\psi+\mu\frac{\partial}{\partial x}\psi+\Sigma_t\psi=\frac{\Sigma_s}{2}\phi+Q$$

Streaming Term — Scattering Term

The time-dependent transport equation is useful for a variety of problems including astrophysics simulations and inertial confinement fusion.

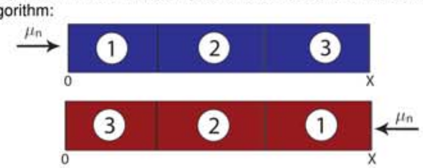
Classification of Time Discretizations

Currently, there were two major classifications of time discretizations:

- Conditionally stable discretizations:
  - Typically fast for a single time step
  - Often scale linearly with processors
  - Require very small Δt for stability
  - Examples include Explicit and Crank-Nicholson
  - More recently developed, Klar's Method has a relaxed stability condition but is limited to highly-diffusive problems
- Unconditionally stable discretizations
  - Unconditionally stable for any Δt
  - Typically require DSA and possibly Krylov solvers
  - Require sweeps, which limits parallel scalability
  - Example: Implicit

Mesh Sweeps

Traditionally, deterministic transport problems have been solved using a mesh-sweep algorithm:



Mesh-sweep algorithms require that each cell be solved in a specific sequence. This limits the parallel efficiency of the transport algorithm.

Additionally, transport methods employing mesh-sweeps typically iterate over the scattering source. This requires diffusion synthetic acceleration and Krylov solvers, further complicating the design, implementation and parallelization of the transport method.

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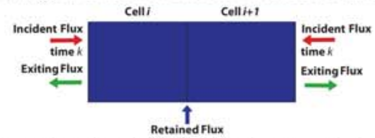
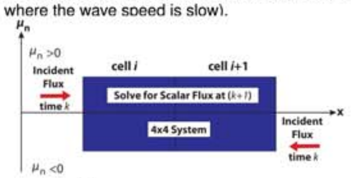
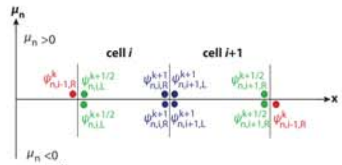
Staggered Block Jacobi Method (SBJ)

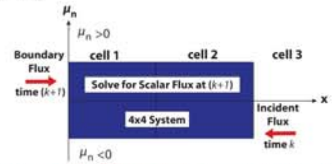
Staggered Block Jacobi Concept: Invert a two-cell block (in 1-D) using incident information lagged to the previous time step. Only angular fluxes on the two-cell interface are retained.

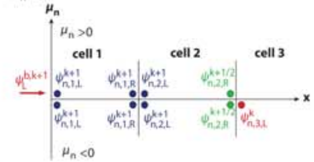


This method should be accurate where the solution does not change greatly during a time step (i.e., where the wave speed is slow).









Mesh-sweeps and the SBJ method are complementary:

Mesh sweeps:
- Optimal for optically thin media
- Hard to parallelize

Staggered Block Jacobi method:
- Optimal for optically thick media
- Easy to parallelize

To observe the stability properties of the SBJ method, consider the thick diffusive limit:

$$\left(\frac{\Delta x}{v\Delta t_{k+1}}+\frac{2D}{\Delta x}+\Sigma_a\Delta x\right)\phi_{i+1/2}^{k+1}=\left(\frac{\Delta x}{v\Delta t_{k+1}}\right)\phi_{i+1/2}^{k}+\left(\frac{D}{\Delta x}\right)\phi_{i-1/2}^{k}+\left(\frac{D}{\Delta x}\right)\phi_{i+3/2}^{k}$$
$$+\frac{\Delta x}{2}\sum_{n=1}^{N}q_{n,i,R}^{k+1}\Delta_n+\frac{\Delta x}{2}\sum_{n=1}^{N}q_{n,i+1,L}^{k+1}\Delta_n$$

This equation is positive and unconditionally stable. However, it does not preserve particle balance. We can restore particle balance on the *domain* (but not *cell-wise*) by rebalancing our angular fluxes:

$$\psi^{k+1,cons}=\gamma^{k+1}\psi^{k+1}$$

where

$$\gamma^{k+1}=\frac{\textit{flux at time } k+\textit{source}+\textit{particles leaving domain}}{\textit{non-conservative flux at }(k+1)+\textit{particles absorbed}+\textit{incident particles}}$$

Using Sweeps to Increase Accuracy

SBJ is **accurate when the particle wave moves less than one cell per time step**. Otherwise, accuracy degrades. We can improve accuracy using a single sweep with a lagged scattering source. We can improve the accuracy of the sweep by stretching the equations:

$$\frac{\epsilon}{v\Delta t_{k+1}}\left(\psi_0^{k+1}-\psi^{k}\right)+\mu_n\frac{d}{dx}\psi_0^{k+1}+\Sigma_t\epsilon\,\psi_0^{k+1}=\frac{1}{2}\left(\frac{\Sigma_t}{\epsilon}-\epsilon\,\Sigma_a\right)\phi^{k}+\epsilon Q^{k+1}$$

The term $\left(\frac{\Sigma_t}{\epsilon}+\frac{\epsilon}{v\Delta t_{k+1}}\right)$ is the inverse of the mean free path of the particle. We want to maximize the mfp, therefore

Thin Limit: $\epsilon=1$ Thick Diffusive Limit: $\epsilon=\sqrt{\Sigma_t v\Delta t_{k+1}}$

We have used

$$\epsilon_i^{k+1}=1.0+\sqrt{v\Delta t_{k+1}\Sigma_{t,i}}\left[1.0-e^{-(0.07v\Delta t_{k+1}\Sigma_{t,i})}\right]$$

Increasing Accuracy with Iterations

We can improve the accuracy of the SBJ scheme using iterations along with sweeps:

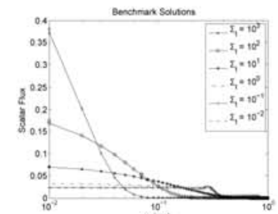
1. Perform a stretched sweep to capture the solution in thin streaming regions
2. Perform a SBJ solve to capture the solution in thick diffusive regimes
3. Use the results from the SBJ as the incoming flux on each block
4. Repeat steps 2 and 3 until the desired accuracy is achieved.

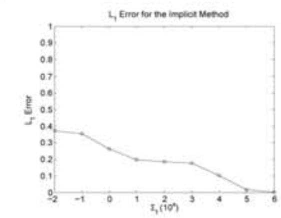
Results

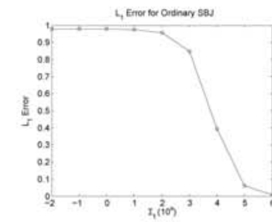
- Mesh width: 1.0 cm
- Number of zones: 100
- $\Sigma_t=\Sigma_s=10^{-2} \dots 10^{6}$
- Q = 1.0 in first cell, Q = 0.0 elsewhere
- v = 1.0 cm/s
- Reflecting boundary on left, vacuum on righ
- Zero initial angular flux
- Δt = 1.0, Benchmark $\Delta t = 10^{-5}$
- t_f=1.0 s

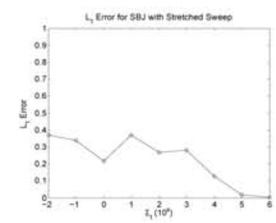
We calculate the relative error using:

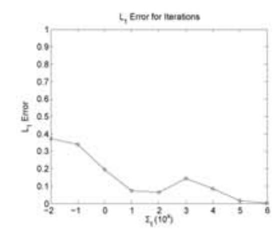
$$L_1\,Error=\frac{\sum_{i=1}^{I}|\phi_i^{Benchmark}-\phi_i|\Delta x_i}{\sum_{i=1}^{I}\phi_i^{Benchmark}\Delta x_i}$$

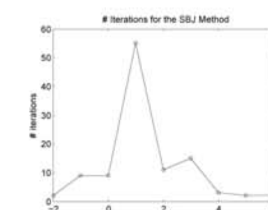












Future Work

We are now applying this work to non-linear thermal radiation transport problems.